Calculation of monthly payments (amortization)

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This document shows step-by-step how the payments for a fixed payment loan is calculated. Given an initial principal amount, P_0, an annual interest rate, R, and a number of monthly payments, n, how do we calculate a constant payment, m, so that after n monthly payments of m each, the principal balance, P_n, will be 0?

To begin with, the monthly interest rate, r is given by

$$r = R/12 \tag{1}$$

At the beginning of the loan time, that is month 0, the current principal is simply the initial principal.

$$P_0 = P_0 \tag{2}$$

At the end of the first month, the balance is the initial principal, plus the interest accrued during that month, minus whatever payment we've made.

$$P_1 = P_0 + P_0 r - m \tag{3}$$

$$P_1 = P_0(1 + r) - m \tag{4}$$

At the end of the second month, the balance is P_1 plus interest accrued (on the slightly-lower balance) minus another monthly payment

$$P_2 = P_1 + P_1 r - m \tag{5}$$

$$P_2 = P_1(1 + r) - m \tag{6}$$

Substituting for P_1 from eq. 4 gives

$$P_2 = (P_0(1 + r) - m)(1 + r) - m \tag{7}$$

$$P_2 = P_0(1 + r)^2 - (1 + r)m - m \tag{8}$$

At the end of month n, the principal balance, P_n, is given by

$$P_n = P_0(1 + r)^n - (1 + r)^{n-1}m - (1 + r)^{n-2}m \ldots - (1 + r)m - m \tag{9}$$

Here, all the terms containing m are a geometric series that can easily be expressed in *sigma* notation:

$$P_n = P_0(1+r)^n - m\sum_{i=0}^{n-1}(1+r)^i \tag{10}$$

The algebraic expansion of the sum of the first n terms of a geometric series, S_n with constant ratio, ρ, goes something like this:

$$S_n = \sum_{i=0}^{n-1} a\rho^i \tag{11}$$

$$S_n = a + a\rho + a\rho^2 \ldots\ + a\rho^{n-2} + a\rho^{n-1} \tag{12}$$

Multiplying both sides by ρ gives

$$\rho S_n = a\rho + a\rho^2 \ldots\ + a\rho^{n-1} + a\rho^n \tag{13}$$

Subtracting eq. 12 from eq. 13 gives

$$\rho S_n - S_n = a\rho^n - a \tag{14}$$

and then solving for S_n

$$S_n(\rho - 1) = a(\rho^n - 1) \tag{15}$$

$$S_n = \frac{a(\rho^n - 1)}{\rho - 1} \tag{16}$$

$$\sum_{i=0}^{n-1} a\rho^i = \frac{a(\rho^n - 1)}{\rho - 1} \tag{17}$$

Substituting eq. 17 in eq. 10 with $a = 1$ and $\rho = (1 + r)$ yields

$$P_n = P_0(1+r)^n - m\frac{(1+r)^n - 1}{(1+r) - 1} \tag{18}$$

$$P_n = P_0(1+r)^n - m\frac{(1+r)^n - 1}{r} \tag{19}$$

If the loan is to be paid in full at month n, that means $P_n = 0$

$$0 = P_0(1+r)^n - m\frac{(1+r)^n - 1}{r} \tag{20}$$

Solving this equation for m

$$m\frac{(1+r)^n - 1}{r} = P_0(1+r)^n \tag{21}$$

$$m = P_0(1+r)^n\frac{r}{(1+r)^n - 1} \tag{22}$$

$$m = \frac{P_0(1+r)^n r}{(1+r)^n - 1} \tag{23}$$